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Duality in fractals

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Abstract. It is shown that the Bethe lattice with coordination number 4 is a dual lattice for Sierpinski's gasket.

1. Introduction

Recently, a great deal of interest has been shown in fractals (Mandelbrot 1977, Suzuki 1983, Rammal and Toulouse 1983). The reason is that numerous realisations of self-similar structures or fractals (i.e. systems with non-integer dimensionality) exist around us (Stapleton *et al* 1980, Alexander and Orbach 1982). On the one hand, irregular systems (in a geometrical sense) which are fractals can describe amorphous materials very well. On the other hand, dynamical systems in N-dimensional phase space can be defined applying the concept of fractal dimension (Farmer 1982).

In this paper we consider fractal systems (structures with non-integer dimensionality). The motivation for our paper is as follows. As fractals can simulate amorphous materials or other systems, irregular in shape, it is interesting to examine their phase transitions and high- and low-temperature behaviours. By placing spins (Ising, Potts, etc) onto the fractal lattice we obtain the model for magnetic amorphous material. The high- and low-temperature behaviour can be determined by applying the duality transformation for the system (Wegner 1971, 1973, Savit 1980). It has been shown that the high-temperature region of one theory maps itself under the duality transformations onto the low-temperature region of another theory (the so-called dual theory, while the previous one is referred to as an original theory). So, using the duality transformation, one can obtain information about the nature of the phase transitions, structures, high- and low-temperature regions of the system, etc. There are many papers concerning the thermodynamical properties of spin systems (Ising, XY, Potts model) on usual integer-dimensional lattices, and on pseudolattices such as the Bethe lattice (Wu 1982). On the contrary, however, few papers deal with the problem of the high- and low-temperature behaviour of spin systems of non-integer-dimensional lattices; for example, there is an analysis of the behaviour of the classical XY and cluster models on the truncated tetrahedral lattice (Dhar 1977), as well as a paper on the phase transition on fractals (Suzuki 1983).

However, the reason for the scarcity of papers on the thermodynamics of fractal lattices is that the dual lattice for non-integer lattices has not been considered hitherto. Our paper aims to derive the dual fractal lattice.

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Because of the importance of Sierpinski's gasket in percolation phenomena (Gefen *et al* 1980), we search for its dual lattice. The construction of the present paper is as follows. In § 2, we propose a formalism needed for solving our problem. In § 3 we present a theorem dealing with the dual lattice for Sierpinski's gasket. Next, in § 4, we present some properties of this dual lattice. The notation is set out in table 1.

Table 1. Notation.

$\{B\}, (\{B^*\})$	the original lattice (the dual lattice)
$\{B\}_i, (\{B^*\}_i^*)$	the lattice in the ith step of iteration (the dual lattice for the ith step of iteration)
C_i	cycle for ith step of iteration
$d, (d')$	dimension of an original fractal (dual fractal)
d_E	dimension of Euclidean space
D	dilution transformation
E	Euclidean space
$f_d, (F_{d'})$	original fractal (dual fractal)
$B_0, (B_0^*)$	site of original lattice (dual lattice)
$\{B_0\}, \{B_{01}, B_{02}, \ldots\}$	set of sites
$B_1, (B_1^*)$	bond of the original lattice (dual lattice)
$\{B_1\}, \{B_{1,1}, B_{1,2}, \ldots\}$	set of bonds
B_2	face of the original lattice
$\lvert\overline{B_{0i}B_{0i+1}}\rvert$	bond of the original lattice
$R_{i,i+1}$	length of bond with the end points B_{0i}, B_{0i+1}
λ	scaling factor
T	duality transformation
$\subset$	symbol of set inclusion
$\otimes$	symbol of a direct product
$\cap$	symbol of a set product
$\cup$	symbol of a set sum

2. Definitions

Let us consider a formalism needed to understand our theorem concerning the fractal lattice.

Mandelbrot (1977) has introduced the concept of fractals. But now what is a fractal lattice? There are a number of definitions of the fractal lattice (Gefen *et al* 1980). We give a definition of the fractal lattice based on these published results. Before this, we consider two transformations, which will be useful in the next part of this paper.

Let us consider the dilution transformation D. If by $\{Z\}$ we define the set of figures (Suzuki 1983):

$$\{Z\} \equiv \{Z_0, Z_1, \ldots,\}$$

then

$$DZ_0 = Z_1, \qquad DZ_1 = Z_2, \qquad \ldots, \qquad DZ_n = Z_{n+1} \tag{1a}$$

or, in another way

$$DZ_0 = \lambda Z_0, \qquad DZ_1 = \lambda Z_1, \qquad \ldots, \qquad DZ_n = \lambda Z_n. \tag{1b}$$

We easily see that

$$D^n Z_0 = D[D^{n-1} Z_0] = \ldots = \underbrace{D[\ldots[}_{n} DZ_0]\ldots] \tag{2}$$

and

$$D^n Z_0 = Z_n.$$

The inverse transformation is

$$D^{-1} Z_i = Z_{i-1} = \lambda^{-1} Z_{i-1} \tag{3}$$

and

$$DD^{-1} Z_i = D^{-1} DZ_i = D[D^{-1} Z_i] = D^{-1}[DZ_i] = Z_i \tag{4}$$

and

$$D[Z_i + Z_j] = DZ_i + DZ_j. \tag{5}$$

Let us now consider the duality transformation (Wegner 1971). Let A be an arbitrary set of figures

$A = \{A_0, A_1, A_2, \ldots,\}$ or spaces, then

$$TA = A^*, \tag{6}$$

where A^* is the dual set of figures (or dual space).

If we take into account the lattice $\{B\}$, then

$$T\{B\} = \{B^*\} \tag{7}$$

and, in particular,

$$TB_2 = B_0^*, \qquad TB_1 = B_1^*. \tag{8}$$

We easily note that

$$DT\{B\} = TD\{B\}_i. \tag{9}$$

We introduce the following definition of the fractal lattice.

Definition 1. A fractal lattice $\{B\}$ is a system of points $B_0 \subset \{B\}$ linked by bonds $B_1 \subset \{B\}$ which satisfy the following criteria

(i) $\{B\}$ is non-invariant under a translation transformation, contrary to Euclidean integer-dimensional lattices,

(ii) $\{B\}$ is invariant under the dilution transformation D, so that

$$D\{B\} = \{B'\} = \lambda\{B\}. \tag{10}$$

This means that each of the elements of $\{B\}$ is rescaled by the same factor λ.

Remark. The fractal lattice $\{B\}$ can be determined recursively. Before giving the definition of the dual fractal lattice (the fundamental definition for this paper), we present a definition of the dual fractal in order to set precise criteria for its existence.

Definition 2. Let f_d be a d-dimensional fractal embedded in the d_E-dimensional Euclidean space $E(d_E = 1, 2, \ldots)$.

We say that $F_{d'}$ is a d'-dimensional dual fractal if

$$E = f_d \cup F_{d'}. \tag{11}$$

The definition of the dual fractal lattice requires introducing the concept of the direct product of two lattices. Using the definition of the direct lattice product given by Dhar (1977) and Mandelbrot (1977), we write the following.

Definition 3. (Dhar 1977, Mandelbrot 1977). Let $\{B\}$ and $\{G\}$ be two fractal lattices, then

$$\{H\} = \{B \times G\}$$

is said to be the direct lattice product if, to each ordered pair of points (B_0, G_0) such that $B_0 \subset \{B\}$, $G_0 \subset \{G\}$, we can assign a unique lattice point $H_0 \subset \{H\}$.

Remark. (Dhar 1977, Mandelbrot 1977). The dimension of $\{H\}$:

$$\dim\{H\} = \dim(\{B\} \times \{G\})$$

is a sum of the dimensions of the lattices $\{B\}$ and $\{G\}$; this means that

$$\dim(\{B\} \times \{G\}) = \dim\{B\} + \dim\{G\}.$$

We easily see that the usual two-dimensional square lattice $\{s\}$ is a direct product of two linear chains $\{L\}$, $\{L'\}$ and its dimensionality is

$$\dim\{s\} = 2 = 1 + 1 = \dim\{L\} + \dim\{L'\}.$$

We now introduce the following.

Definition of the dual fractal lattice. The set of points $B_0^* \subset \{B^*\}$, linked by the bonds $B_1^* \subset \{B^*\}$, is said to be a dual fractal lattice $\{B^*\}$ for the original fractal lattice $\{B\}$ if it satisfies the criteria (see also figure 1):

(i) $\{B^*\}$ is translationally non-invariant,

(ii) $\{B^*\}$ is dilution invariant,

(iii) $\dim\{E\} = \dim\{B^*\} + \dim\{B\}$,

(iv) each of the points $B_0^* \subset \{B^*\}$ is situated at the centre of a face $B_2 \subset \{B\}$ in the original lattice,

$$TB_2 = B_0^* \qquad \text{(Wegner 1971)},$$

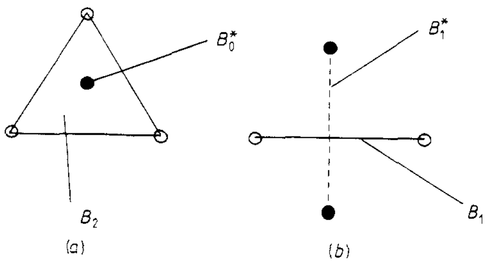


Figure 1. (*a*) The face B_2 of the original lattice $\{B_1\}$ and its dual point B_0^* in the dual lattice $\{B^*\}$. (*b*) The bond B_1 of the original lattice $\{B\}$ and its dual bond $B_1^* \subset \{B^*\}$.

(v) each of the nearest-neighbour points B^*_{0i}, $B^*_{0i+1} \subset \{B^*\}$ $(i = 1, 2, \ldots,)$ of the dual lattice is linked by the bonds B^*_1 in such a manner that

$$B^*_1 = \min\{b^*_1\},$$

where $\{b^*_1\}$ is the set of all possible links which connect different points in $\{B^*\}$.

(vi) each of the bonds B^*_1 is crossed by a corresponding bond $B_1 \subset \{B\}$ in such a manner that

$$B^*_{1i} \perp B_{1j} \qquad (i, j = 1, 2, \ldots),$$

meaning that for every i there exists such a j that (i) is fulfilled.

(vii) $TB^*_{1i} = B_{1j}$ (Wegner 1971).

Remark. It is easily seen that

$$B_0 \cap B_2 = B^*_0, \qquad B^*_1 \cap B_1 = x,$$

where x is an intersection point of both bonds B^*_1, B_1.

3. The dual fractal lattice for the Sierpinski gasket

Let us recall what Sierpinski's gasket is. In figure 2 we present the first three stages of its construction. In the first stage there is a triangle, in the second stage we have three triangles with the central triangle eliminated, and so on.

Let us consider some useful notations. It is very convenient with regard to the results, to define the lattice by introducing the concept of graphs. These graphs construct our lattice and are built of points and bonds. Each of the graphs is described by the coordinatión number z (Essam and Fischer 1970).

Taking into account the definition of the dual fractal lattice, we can construct the dual lattice for Sierpinski's gasket. The result is presented by the following theorem.

Theorem. The dual fractal lattice for the Sierpinski gasket is given by Bethe lattice with the coordination number $z = 4$.

Proof. This consists of two parts. Firstly, we show that the lattice cannot include any cycles, and this means that $\{B^*\}$ is a Cayley tree. At this point we consider the invariance of the lattice with cycle under the D transformation and T transformation. If the lattice with the cycle is non-invariant under D and T, it cannot be a dual lattice. Next, we show that using the D and T transformation operators, the coordination number is $z = 4$.

(1) Suppose that there is a cycle $\{C_i\}$ on the ith stage of the iteration of the lattice $\{B^*\}$. Let us check whether C_i is D- and T-invariant.

Let P_j be a perimeter of the graph $\{B^*\}$ without cycles;

$$P_j = \sum_{k=1}^{j} R_{k,k+1} = 3(R_{0,1} + R_{1,2} + \ldots + R_{j,j+1}),$$

where $R_{k,k+1} = |B^*_{0k} B^*_{0k+1}|$ is the distance between the two nearest-neighbour points B^*_{0k}, $B^*_{0k+1} \subset \{B^*\}$.

Now

$$DR_{k,k+1} = R_{k+1,k+2} = \lambda R_{k,k+1},$$

and then

$$P_k = 3R_{0,1} \sum_{p=0}^{k} \lambda^P,$$

and

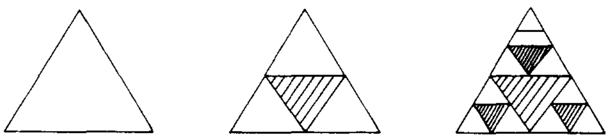
$$DP_k = P_{k+1} = 3\lambda R_0 \sum_{P=0}^{k+1} \lambda^P$$

from equations (2)-(8). So the lattice $\{B\}_k$ is D-invariant. Let us check whether P_j is D-invariant.

$$P_j \equiv P_j(C_j) = 3R_0 \sum_{P=0}^{j} \lambda^j + R_{j+1,j+1}$$

where the cycle C_j is built of the bonds (figure 3):

$$\overline{|B^*_{0j} B^*_{0j+1}|}, \qquad \overline{|B^*_{0j} B^{*\prime}_{0j+1}|}, \qquad \overline{|B^*_{0j+1} B^{*\prime}_{0j+1}|}.$$



(a) (b) (c)

Figure 2. The Sierpinski's gasket and (a) the first stage of iteration, (b) the second stage and (c) the third stage of iteration. The shaded triangles are eliminated.

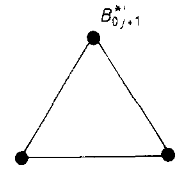


Figure 3. The cycle C_j.

Now

$$DP_j = P_{j+1} + DR_{j+1,j+1}.$$

However,

$$DR_{j+1,j+1} \neq \lambda^{j+1} R_{01}$$

so the perimeter P_j of the lattice $\{B^*\}$ with the cycle C_j is non-invariant under D transformation.

Let us check whether the closing sector of the cycle C_j i.e. $\overline{|B^*_{0j+1} B^*_{0j+1}|}$ is T-invariant.

The dual lattice on the jth step of iteration, including the cycle C_j, can be described as follows:

$$\{B^*\}_j = \{\{B^*\}_{j-1} \overline{|B^*_{0j+1} B^{*\prime}_{0j+1}|}, \overline{|B^*_{0j} B^*_{0j+1}|}, \overline{|B^*_{0j} B^{*\prime}_{0j+1}|}, \overline{|B^*_{0j} B^{*\prime}_{0j+1}|}\}.$$

Acting with the operator T on $\{B^*\}_j$, we obtain

$$T\{B^*\}_j = \{T\{B^*\}_{j-1}, T\overline{|B^*_{0j+1} B^{*\prime}_{0j+1}|}, T\overline{|B^*_{0j} B^*_{0j+1}|}, T\overline{|B^*_{0j} B^{*\prime}_{0j+1}|}, T\overline{|B^*_{0j} B^*_{0j+1}|}\}.$$

From the definition of the dual lattice

$$T\{B^*\}_{j-1} = \{B\}_{j-1},$$

$$T\overline{|B^*_{0j} B^{*\prime}_{0j+1}|} = \overline{|B_{0j} B'_{0j+1}|},$$

$$T\overline{|B^*_{0j} B^*_{0j+1}|} = \overline{|B_{0j} B_{0j+1}|},$$

$$T\overline{|B^*_{0j} B^{*\prime\prime}_{0j+1}|} = \overline{|B_{0j} B_{0j+1}|},$$

then

$$T\{B^*\}_j = \{\{B\}_{j+1}, \overline{T|B^*_{0j+1}B^{*\prime}_{0j+1}|}\}.$$

However, (see also figure 4),

$$\overline{T|B^*_{0j+1}B^{*\prime}_{0j+1}|} = \overline{|B^*_0 B''_{01}|}$$

and

$$\overline{|B^*_0 B''_{01}|} \not\subset \{B^*\},$$

$$\overline{|B^*_0 B''_{01}|} \not\subset \{B\},$$

$$\overline{\{B^*_0 B''_{01}|} \subset \{B^*\} \cap \{B\}.$$

This means that the closing sector of the cycle C_j is non-invariant under T transformation, and that the lattice $\{B^*\}$ with this cycle is T-non-invariant also.

Then $\{B^*\}$ is a Cayley tree.

(2) Acting on a $\{B\}_1$ lattice with the operators T and D we obtain in the second stage of iteration:

$$TD\{B\}_2 = \{(B^*_{01}, B^{*\prime}_{01}, B^{*\prime\prime}_{01})_1, (B^*_{01}, B^{*\prime}_{01}, B^{*\prime\prime}_{01})_2, (B^*_{01}B^{*\prime}_{01}B^{*\prime\prime}_{01})_3\}.$$

and acting with D^{-1} on $\otimes$ we obtain

$$D^{-1}TD\{B\}_2 = \{B^*_{01}, B^*_{02}, B^*_{03}\}.$$

In this manner we learn that the points $(B^*_{01}, B^{*\prime}_{01}, B^{*\prime\prime}_{01})_1$ are linked with the point B^*_{01}, the points $(B^*_{01}B^{*\prime}_{01}B^{*\prime\prime}_{01})_2$ with B^*_{02}, and $(B^*_{01}B^{*\prime}_{01}B^{*\prime\prime}_{01})_3$ with B^*_{03}. Now

$$DD^{-1}TD\{B\}_2 = \{B^*_0\}.$$

This means that the points $B^*_{01}B^*_{02}B^*_{03}$ are linked with the points B^*_0. Then the coordination number $z = 4$. $\{B^*\}$ is a Bethe lattice.

4. The properties of the dual fractal lattice

Let us analyse the dual lattice obtained above. It is defined recursively (figure 5). Obviously, the number of vertices in that lattice on the rth step of iteration is

$$V_r = 4 \cdot 3^r + \sum_{i=0}^{r-1} 3^i$$

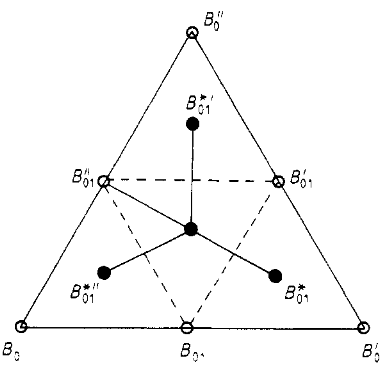


Figure 4. The closing sector $\overline{|B^{*\prime\prime}_{01}B^{*\prime}_{01}|}$ and its dual sector $|B''_{01}B^*_0|$.

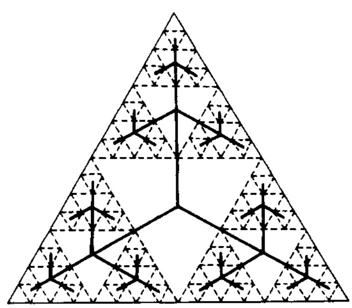


Figure 5. The Sierpinski gasket and its dual Bethe lattice with $z = 4$.

and the number of bonds is

$$B_r = 3^r\left(4 + \sum_{i=1}^{r-1} 3^{-i}\right).$$

5. Conclusion

We have proposed a dual fractal lattice for Sierpinski's gasket. It turns out that it is a Bethe lattice with the coordination number $z = 4$.

This result is very promising because of the existence of a great number of papers about the properties of the Bethe lattice. Thus, the investigation of the thermodynamical properties of the fractal lattice is made easier.

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